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Immersions of two-manifolds in Euclidean spaces

Introduction. S. S. Chern and R. K. Lashof have proved the following

Theorem [1]. *If the n-sphere S^n is immersed in the Euclidean $(n+N)$-space E^{n+N} with minimal total curvature, then the image of S is a convex hypersurface in a hyperplane $E^{n+1}\subset E^{n+N}$ and the immersion is a diffeomorphism.*

With the aid of a method developed in [1] we obtain in the present note a similar result concerning immersions of orientable two-manifolds in the Euclidean n-space. We prove that if the two-manifold immersed has minimal total curvature and satisfies some additional conditions (conditions (A)), then the manifold immersed is a surface in a three-plane $E^3\subset E^{n+2}$, $n\geqslant 1$. The Killing imbedding of the torus in the four-space shows that the conditions (A) are essential.

1. The moving frame. Let

$$X\to X'=AX+b$$

denote an element of the Euclidean group $E(n+2, R)$ over the reals R, of transformations of the Euclidean space E^{n+2}. Denoting by A^T the matrix transposed to A we have

$$X^T=(x_1, x_2, \ldots, x_{n+2})\in E^{n+2},$$

$b^T=(b_1, b_2, \ldots, b_{n+2})$ is the displacement vector,

$$A=\|a_{ij}\|, \quad 1\leqslant i, j\leqslant n+2,$$

is an orthogonal matrix.

Suppose that

$$Y'=AY+b=\bar{A}X+\bar{b}.$$

Thus we have

$$Y'-X'=A(Y-X), \quad Y'-X'=(\bar{A}-A)X+\bar{b}-b.$$

Replacing the finite increments by infinitesimal ones we obtain

$$A\cdot\delta X=dA\cdot X+db.$$

The solution of the last equation with regard to the δx_i's takes the form

$$\delta X=A^T dA\cdot X+A^T db. \tag{1}$$

The elements $\omega_i = a_{ji} db_j$, $\omega_{ij} = a_{ki} da_{kj}$ of the matrices $A^T db$, $A^T dA$ are the components of the infinitesimal displacement of the frame

(2) $$b^T e_1 e_2 \ldots e_{n+2}, \qquad e_i = (a_{1i}, a_{2i}, \ldots, a_{n+2,i})$$

onto the frame

$$b^T + db^T e_1 + de_1 \ldots e_{n+2} + de_{n+2}$$

with respect to the frame (2).

Indeed, the transformation

$$b^T e_1 e_2 \ldots e_{n+2} \to b^T + db^T e_1 + de_1 \ldots e_{n+2} + de_{n+2}$$

takes the form

(3) $$Y^* + \delta Y^* = (A + dA) X^* + b + db.$$

In the preceding formula X^* denotes the matrix calculated from the equation $Y^* = AX^* + b$. In the coordinate system associated with the frame (2) the formula (3) has the form

$$A \cdot \delta Y = dA \cdot X + db$$

if $Y^* = AY + b$, $X^* = AX + b$ denote the change of the coordinates. We see that the last formula is equivalent to (1).

We have

(4) $$\omega_i = e_i db^T, \qquad \omega_{ij} = e_i de_j.$$

Exterior differentiation of (4) leads to the equations of structure of the Euclidean group

(5) $$d\omega_j = \omega_i \wedge \omega_{ji}$$
$$d\omega_{ij} = \omega_{kj} \wedge \omega_{ik}$$

and from $e_i e_j = \delta_{ij}$ we obtain

$$\omega_{ij} + \omega_{ji} = 0.$$

2. Immersion. The mapping of an orientable closed manifold

$$x : M^2 \to E^{n+2}$$

is said to be an immersion if it belongs to the class C^∞ and if the induced mapping of tangent planes is one to one. From now on we assume that $x: M^2 \to E^{n+2}$ is an immersion and that M^2 is a closed, orientable C^∞ manifold.

In order to obtain the equations of structure of the surface $x(M^2)$ let us consider those elements of $E(n+2, R)$ for which $b^T \in x(M^2)$ and e_1, e_2 are tangent to $x(M^2)$ at $b^T(p)$, $p \in M^2$. Hence using (4) we obtain

$$db^T(p) = e_1 \omega_1 + e_2 \omega_2, \qquad \omega_3 = \omega_4 = \ldots = \omega_{n+2} = 0,$$

and it follows from (5)

$$\omega_1 \wedge \omega_{1r} + \omega_2 \wedge \omega_{2r} = 0, \qquad r = 3, 4, \ldots, n+2, \tag{6}$$

$$d\omega_\beta = \omega_\alpha \wedge \omega_{\alpha\beta}, \qquad \alpha, \beta, \gamma, \delta = 1, 2,$$

$$d\omega_{\alpha\beta} = \omega_{\gamma\beta} \wedge \omega_{\alpha\gamma} + \Omega_{\alpha\beta}, \qquad \Omega_{\alpha\beta} = \omega_{r\beta} \wedge \omega_{r\alpha}.$$

CARTAN's lemma applied to (6) yields the formulas

$$\begin{aligned} \omega_{1r} &= A_{r11}\,\omega_1 + A_{r12}\,\omega_2, \\ \omega_{2r} &= A_{r21}\,\omega_1 + A_{r22}\,\omega_2, \end{aligned} \qquad A_{r12} = A_{r21}. \tag{7}$$

If we substitute the last formulas in the curvature forms $\Omega_{\alpha\beta}$ we obtain

$$\begin{aligned} \Omega_{\alpha\beta} = \omega_{r\beta} \wedge \omega_{\alpha r} &= -A_{r\beta\gamma} A_{r\alpha\delta}\,\omega_\gamma \wedge \omega_\delta \\ &= (A_{r\alpha 1} A_{r\beta 2} - A_{r\alpha 2} A_{r\beta 1})\,\omega_1 \wedge \omega_2 = R_{\alpha\beta 12}\,\omega_1 \wedge \omega_2. \end{aligned}$$

The scalar coefficient R_{1212} is the GAUSS curvature $K(p)$ of the surface $x(M^2) \subset E^{n+2}$

$$K(p) = \sum_{r=3}^{n+2} \det(A_{r\alpha\beta}). \tag{8}$$

The immersion $x: M^2 \to E^{n+2}$ defines a manifold which is denoted by B_ν and is called the normal bundle of M^2. We define

$$B_\nu = \{(p, \nu) \mid \nu \cdot dx = 0\}, \tag{9}$$

where ν denotes an arbitrary unit vector in E^{n+2}, i.e. B_ν is the set of such pairs (p, ν) that ν is orthogonal to the surface $x(M^2)$ at $x(p)$. In B_ν we introduce the volume element

$$d\tau_{n+1} = dV_2 \wedge d\sigma_{n-1}, \tag{10}$$

$$dV_2 = \omega_1 \wedge \omega_2, \qquad d\sigma_{n-1} = \omega_{n+2,3} \wedge \omega_{n+2,4} \wedge \ldots \wedge \omega_{n+2,n+1}.$$

Thus dV_2 is a surface element of M^2 induced by the immersion x and $d\sigma_{n-1}$ is the volume element of the $(n-1)$-dimensional unit sphere.

The volume element of the $(n+1)$-dimensional unit sphere S^{n+1} described by e_{n+2} can be written in the form

$$d\sigma_{n+1} = \omega_{n+2,1} \wedge \omega_{n+2,2} \wedge \ldots \wedge \omega_{n+2,n+1}. \tag{11}$$

We have from (7), (10) and (11)

$$d\sigma_{n+1} = \det(A_{n+2\alpha\beta})\,dV_2 \wedge d\sigma_{n-1}. \tag{12}$$

The function $L(p, e_{n+2}) = \det(A_{n+2\alpha\beta})$ will be named the LIPSCHITZ-KILLING curvature of B_ν. In view of (10) we see that $L(p, \nu)$ is the Jacobian determinant of the mapping $B_\nu \to S^{n+1}$ defined by

$$(p, \nu) \to \nu, \qquad (p, \nu) \in B_\nu.$$

The following inequality is valid [2]

$$\int_{B_v} |L(p, v)|\, d\tau_{n+1} \geqslant c_{n+1} \sum_{k=0}^{2} p_k(M^2), \tag{13}$$

where $p\,(M^2)$ denotes the k-th BETTI number of M^2 and c_{n+1} is the volume of S^{n+1}.

DEFINITION. We say that x is an immersion with minimal total curvature if in (13) the equality holds.

Assume that there exists such a point $p_0 \in M^2$ for which the following conditions are satisfied:

$$L(p_0, v) \geqslant 0, \qquad v \in S^{n-1}(p_0)$$

$$K(p_0) = \sum_{r=3}^{n+2} L(p_0, e_r) > 0 \qquad \text{(see (8))}.$$

A point $p_0 \in M^2$ is named a critical point of the scalar function $v \cdot x(p)$ (v is fixed) if $(p_0, v) \in B_v$. The point p_0 is named a critical non-degenerated point if it is a critical point and moreover $L(p_0, v) \neq 0$. From the equality

$$e_{n+2} \cdot d^2x = -de_{n+2} \cdot dx = A_{n+2\alpha\beta}\, \omega_\alpha \omega_\beta \tag{14}$$

we see that the determinant of the quadratic form (14) is equal to $L(p, e_{n+2})$. A non-degenerated critical point p_0 is said to be of index λ if the maximal rank of such subspaces of the tangent space for which the second quadratic form (14) take negative values is equal to λ. The number $m_\lambda(M^2, v \cdot x(p))$ $(\lambda = 0, 1, 2)$ of critical points of index λ satisfies for almost every $v \in S^{n+1}$ the MORSE inequalities

$$m_\lambda(M^2, v \cdot x(p)) \geqslant p_\lambda(M^2). \tag{15}$$

If in (15) the equality holds for almost every v, then this is equivalent to say that x is an immersion with minimal total curvature. (For detailes see [2, 3]).

THEOREM. *If $x: M^2 \to E^{n+2}$ is an immersion with minimal total curvature and if it satisfies (A), then the surface $x(M^2)$ is a subset of a three-space $E^3 \subset E^{n+2}$.*

Proof. By $p_0 \in M^2$ we denote this point for which the conditions (A) are satisfied. Let $T(p_0, e_{n+2})$ denote the $(n+1)$-dimensional hyperplane which is tangent to the surface $x(M^2)$ at $x(p_0)$ and is orthogonal to e_{n+2}. Suppose that $x(M^2)$ is contained in no $(n+1)$-dimensional hyperplane. Then we may assume that e_{n+2} is chosen in such a manner that there exist two points $x(q_1)$, $x(q_2)$ which lie on different sides of $T(p_0, e_{n+2})$. For

$$v = e_{n+1} \cos\varphi + e_{n+2} \sin\varphi \tag{16}$$

we have

$$v \cdot d^2x = e_{n+1} \cdot d^2x \cos\varphi + e_{n+2} \cdot d^2x \sin\varphi.$$

Hence it follows by (7)

$$v \cdot d^2x = (A_{n+1\alpha\beta} \cos\varphi + A_{n+2\alpha\beta} \sin\varphi)\, \omega_\alpha \omega_\beta$$

and therefore we have

(17) $$L(p_0, v) = L(p_0, e_{n+1})\cos^2\varphi + L(p_0, e_{n+2})\sin^2\varphi.$$

If $L(p_0, e_{n+2})=0$, then it follows from (A) that e_{n+1} can be chosen in such a manner that $L(p_0, e_{n+1})>0$. Therefore we have from (16) that in each neighbourhood of e_{n+2} there exist vectors (namely such of the form (16)) for which

(18) $$L(p_0, v)>0.$$

If the neighbourhood considered above is sufficiently small, then the points $x(q_1)$, $x(q_2)$ remain at different sides of the tangent hyperplane $T(p_0, v)$. Let us consider the critical points of even index, i.e. points $p \in M^2$ such that $(p, v) \in B_v$ and $L(p, v)>0$. Since v is orthogonal to $x(M^2)$ at $x(p_0)$, we see in view of (18) that p_0 is a critical point of even index. Besides p_0 there exist at least two farther critical points $p_1, p_2 \in M^2$ of $v \cdot x(p)$ such that both, $x(p_1)$ and $x(p_2)$ are situated on different sides of $T(p_0, v)$, namely those which correspond to the maximum and minimum of $v \cdot x(p)$. If either p_1 or p_2 is a critical degenerated point (i.e. either $L(p_1, v)=0$ or $L(p_2, v)=0$), then from SARD theorem (see [1, 3]) it follows that in each neighbourhood of v there exist such vectors v' for which the function $v' \cdot x(p)$ has only non-degenerated critical points. If v' is sufficiently near to v, then the function $v' \cdot x(p)$ has also at least three distinct critical points, all non-degenerated and of even index, say p_0', p_1', p_2'. As before, p_1', p_2' correspond to the maximum and minimum of $v' \cdot x(p)$ and the fact that p_0' is a critical point of even index follows from the continuity of $L(p, v)$ and (18). Hence there exist three disjoint neighbourhoods $B(p_i', v') \subset B_v (i=0, 1, 2)$ of $(p_i', v') \in B_v$ such that if $(p_i'', v'') \in B(p_i', v')$, then $L(p_i'', v'')>0$. But if x is an immersion with minimal total curvature, then the equalities

$$m_0(M^2, v \cdot x(p)) = p_0(M^2) = 1 = p_2(M^2) = m_2(M^2, v \cdot x(p))$$

hold for almost every $v \in S^{n+1}$ and it follows that the function $v \cdot x(p)$ has exactly two points of even index. Therefore the assumption that $x(M^2)$ does not lie in $E^{n+1} \subset E^{n+2}$ leads to a contradiction. Hence

(19) $$x(M^2) \subset E^{n+1} \subset E^{n+2}, \quad n \geqslant 2.$$

Denote by η a unit vector orthogonal to the hyperplane E^{n+1}. From (19) we have that each $p \in M^2$ is a critical degenerated point of $\eta \cdot x(p)$. Therefore the vector v' considered above is distinct from η. Let v denote the orthogonal projection of v' on E^{n+1}. Then we have

$$v' \cdot d^2x = \eta d^2x + \tilde{v} \cdot d^2x, \quad \eta b = \tilde{\eta}, \quad b \neq 0.$$

From (19) we have $d^2x \in E^{n+1}$ and therefore

(20) $$v' \cdot d^2x = \tilde{v} \cdot d^2x.$$

Since $L(p, v')$ is a determinant of the quadratic form on the left-hand side of (20), we have

(21) $$L(p_0, \bar{v})>0,$$

$\bar{v}=a\tilde{v}$, $a>0$ is such that $\bar{v}$ is a unit vector.

Now for a fixed $p \in M^2$ the vector v orthogonal to η runs through an $(n-2)$-dimensional sphere $S^{n-2}(p) \subset S^{n-1}(p)$ and therefore

$$L(p_0, v) \geqslant 0, \quad (p_0, v) \in B'_v, \tag{22}$$

where B'_v denotes the normal bundle of the immersion x considered as an immersion in E^{n+1}. Comparing (21) and (22) we see that the immersion $x: M^2 \to E^{n+1}$ satisfies the conditions (A).

Moreover, since the numbers $m_\lambda(M^2, v \cdot x(p))$ remain the same if v runs through E^{n+1}, $x: M^2 \to E^{n+1}$ is an immersion with minimal total curvature.

The described constructions ends after $(n-2)$ steps and we get $x(M^2) \subset E^3$, q.e.d.

The conditions (A) are essential. This may be verified by examining the following

EXAMPLE. Let $x: T^2 \to E^4$ denote the KILLING imbedding of the torus defined by the formulas

$$\begin{aligned} x_1 &= \cos u, \quad x_2 = \sin u, \quad 0 \leqslant u < 2\pi, \\ x_3 &= \cos v, \quad x_4 = \sin v, \quad 0 \leqslant v < 2\pi, \end{aligned} \tag{23}$$

Since the GAUSS curvature of the imbedded torus is identically zero, the conditions (A) are not satisfied.

We must verify that (23) is an imbedding with minimal total curvature. We have

$$dx = (-\sin u\, du, \cos u\, du, -\sin v\, dv, \cos v\, dv).$$

The vector v normal to $x(T^2)$ at $x(u_0, v_0)$ takes the form

$$v = (a \cos u_0, a \sin u_0, b \cos v_0, b \sin v_0), \quad a^2 + b^2 = 1.$$

Hence

$$v \cdot dx = (-\cos u_0 \sin u + \sin u_0 \cos u) a\, du + (-\cos v_0 \sin v + \sin v_0 \cos v)\, b\, dv.$$

From the last formula we have that $v \cdot dx = 0$ for the points

$$(u_0, v_0), \quad (u_0 + \pi, v_0), \quad (u_0 + \pi, v_0 + \pi), \quad (u_0, v_0 + \pi), \quad a, b \neq 0.$$

If either a or b is zero, then v runs through the sphere $S^2 \subset S^3$ and hence through a set of measure zero in S^3. Hence for almost every v the function $v \cdot x(p)$ has exactly four critical points and therefore x is an imbedding with minimal total curvature.

REFERENCES

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MAREK ROCHOWSKI

ZANURZANIA DWUWYMIAROWYCH ROZMAITOŚCI W PRZESTRZENIACH EUKLIDESOWYCH

Streszczenie

W pracy zostało udowodnione następujące uogólnienie twierdzenia S. S. CHERNA i R. K. LASHOFA [1] dla $n = 2$.

Twierdzenie. Jeśli $x: M^2 \to E^{n+2}$ jest zanurzeniem z minimalną krzywizną całkowitą i spełnia warunki (A), to powierzchnia $x(M^2)$ jest podzbiorem przestrzeni euklidesowej trójwymiarowej $E^3 \subset E^{n+2}$.

Warunki (A) mają następujący sens geometryczny: istnieje punkt $x(p_0) \in x(M^2)$ o takiej własności, że rzut pewnego otoczenia tego punktu na powierzchni $x(M^2)$ w przestrzeń trójwymiarową rozpiętą na wektorach stycznych do powierzchni $x(M^2)$ w punkcie $x(p_0)$ i na dowolnym wektorze normalnym ma krzywiznę Gaussa nieujemną oraz istnieje taki wektor normalny dla którego ta krzywizna jest dodatnia.

W pracy podany jest przykład, który wskazuje, że warunki (A) są istotne.

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